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18.175 Theory of Probability
Fall 2008

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Section 11

Lindeberg's CLT. Levy's Equivalence Theorem. Three Series Theorem.

Instead of considering i.i.d. sequences, for each $n \geq 1$ we will consider a vector $(X_1^n, \ldots, X_n^n)$ of independent r.v., not necessarily identically distributed. This setting is called *triangular arrays* because the entire vector may change with n.

Theorem 25 *Consider a vector* $(X_i^n)_{1\leq i\leq n}$ *of independent r.v.s such that*

$$\mathbb{E}X_i^n = 0, \ \ \mathrm{Var}(S_n) = \sum_{i\leq n} \mathbb{E}(X_i^n)^2 = 1.$$

Suppose that the following Lindeberg's condition *is satisfied:*

$$\sum_{i=1}^{n} \mathbb{E}(X_i^n)^2 \mathrm{I}(|X_i^n| > \varepsilon) \to 0 \quad \text{as } n \to \infty \quad \text{for all } \varepsilon > 0. \tag{11.0.1}$$

Then $\mathcal{L}\Big(\sum_{i\leq n} X_i^n\Big) \to \mathcal{N}(0,1)$.

Proof. First of all, $\Big\{\mathcal{L}\Big(\sum_{i\leq n} X_i^n\Big)\Big\}$ is uniformly tight, because by Chebyshev's inequality

$$\mathbb{P}\Big(\Big|\sum_{i\leq n} X_i^n\Big| > M\Big) \leq \frac{1}{M^2} \leq \varepsilon$$

for large enough M. It remains to show that the characteristic function of S_n coverges to $e^{-\frac{\lambda^2}{2}}$. For simplicity of notations let us omit the upper index n and write X_i instead of X_i^n. Since,

$$\mathbb{E}e^{i\lambda S_n} = \prod_{i\leq n} \mathbb{E}e^{i\lambda X_i}$$

it is enough to show that

$$\log \mathbb{E}e^{i\lambda S_n} = \sum \log\Big(1 + \big(\mathbb{E}e^{i\lambda X_i} - 1\big)\Big) \to -\frac{\lambda^2}{2}. \tag{11.0.2}$$

It is an easy exercise to prove, by induction on m, that for any $a \in \mathbb{R}$,

$$\Big|e^{ia} - \sum_{k\leq m} \frac{(ia)^k}{k!}\Big| \leq \frac{|a|^{m+1}}{(m+1)!}. \tag{11.0.3}$$

(Just integrate this inequality to make the induction step.) Using this for $m = 1$,

$$\Big|\mathbb{E}e^{i\lambda X_i} - 1\Big| = \Big|\mathbb{E}e^{i\lambda X_i} - 1 - i\lambda\mathbb{E}X_i\Big| \leq \frac{\lambda^2}{2}\mathbb{E}X_i^2 \leq \frac{\lambda^2}{2}\varepsilon^2 + \frac{\lambda^2}{2}\mathbb{E}X_i^2\mathrm{I}(|X_i| > \varepsilon) \leq \lambda^2\varepsilon^2 \leq \frac{1}{2} \tag{11.0.4}$$

for large n by (11.0.1) and for small enough ε. Using the expansion of $\log(1+z)$ it is easy to check that

$$\Big|\log(1+z) - z\Big| \leq |z|^2 \text{ for } |z| \leq \frac{1}{2}$$

and, therefore, we can write

$$\begin{aligned}\sum_{i\leq n}\Big|\log\big(1 + (\mathbb{E}e^{i\lambda X_i} - 1)\big) - \big(\mathbb{E}e^{i\lambda X_i} - 1\big)\Big| &\leq \sum_{i\leq n}\big|\mathbb{E}e^{i\lambda X_i} - 1\big|^2 \leq \sum_{i\leq n}\frac{\lambda^4}{4}\big(\mathbb{E}X_i^2\big)^2 \\ &\leq \frac{\lambda^4}{4}\big(\max_{i\leq n}\mathbb{E}X_i^2\big)\sum_{i\leq n}\mathbb{E}X_i^2 = \frac{\lambda^4}{4}\max_{i\leq n}\mathbb{E}X_i^2 \to 0\end{aligned}$$

because, as in (11.0.4),

$$\mathbb{E}X_i^2 \leq \varepsilon^2 + \mathbb{E}X_i^2\mathrm{I}(|X_i| > \varepsilon) \to 0$$

for large n by (11.0.1) and for $\varepsilon \to 0$. Finally, to show (11.0.2) it remains to show that

$$\sum_{i\leq n}\big(\mathbb{E}e^{i\lambda X_i} - 1\big) \to -\frac{\lambda^2}{2}.$$

Using (11.0.3) for $m = 1$, on the event $\big\{|X_i| > \varepsilon\big\}$,

$$\Big|e^{i\lambda X_i} - 1 - i\lambda X_i\Big|\mathrm{I}\big(|X_i| > \varepsilon\big) \leq \frac{\lambda^2}{2}X_i^2\mathrm{I}\big((|X_i| > \varepsilon\big)$$

and, therefore,

$$\Big|e^{i\lambda X_i} - 1 - i\lambda X_i + \frac{\lambda^2}{2}X_i^2\Big|\mathrm{I}\big(|X_i| > \varepsilon\big) \leq \lambda^2 X_i^2\mathrm{I}\big(|X_i| > \varepsilon\big).$$

Using (11.0.3) for $m = 2$, on the event $\big\{|X_i| \leq \varepsilon\big\}$,

$$\Big|e^{i\lambda X_i} - 1 - i\lambda X_i + \frac{\lambda^2}{2}X_i^2\Big|\mathrm{I}\big(|X_i| \leq \varepsilon\big) \leq \frac{\lambda^3}{6}|X_i|^3\mathrm{I}\big(|X_i| \leq \varepsilon\big) \leq \frac{\lambda^3\varepsilon}{6}X_i^2.$$

Combining the last two equations and using that $\mathbb{E}X_i = 0$,

$$\Big|\mathbb{E}e^{i\lambda X_i} - 1 + \frac{\lambda^2}{2}\mathbb{E}X_i^2\Big| \leq \lambda^2\mathbb{E}X_i^2\mathrm{I}\big(|X_i| > \varepsilon\big) + \frac{\lambda^3\varepsilon}{6}\mathbb{E}X_i^2.$$

Finally,

$$\begin{aligned}\Big|\sum_{i\leq n}\big(\mathbb{E}e^{i\lambda X_i} - 1\big) + \frac{\lambda^2}{2}\Big| &= \Big|\sum_{i\leq n}\big(\mathbb{E}e^{i\lambda X_i} - 1\big) + \frac{\lambda^2}{2}\sum_{i\leq n}\mathbb{E}X_i^2\Big| \\ &\leq \frac{\lambda^3\varepsilon}{6} + \lambda^2\sum\mathbb{E}X_i^2\mathrm{I}(|X_i| > \varepsilon) \to 0\end{aligned}$$

as $n \to \infty$ (using Lindeberg's condition) and $\varepsilon \to 0$.

□

Lemma 26 *If $\mathbb{P}, \mathbb{Q}$ are distributions on $\mathbb{R}$ such that $\mathbb{P} * \mathbb{Q} = \mathbb{P}$ then $\mathbb{Q}(\{0\}) = 1$.*

Proof. Let us define

$$f_P(t) = \int e^{itx} d\mathbb{P}(x),\ \ f_Q(t) = \int e^{itx} d\mathbb{Q}(x).$$

The condition $\mathbb{P} * \mathbb{Q} = \mathbb{P}$ implies that $f_P(t) f_Q(t) = f_P(t)$. Since $f_P(0) = 1$ and $f_P(t)$ is continuous, for small enough $|t| \leq \varepsilon$ we have $|f_P(t)| > 0$ and, as a result, $f_Q(t) = 1$. Since

$$f_Q(t) = \int \cos(tx) d\mathbb{Q}(x) + i \int \sin(tx) d\mathbb{Q}(x)$$

for $|t| \leq \varepsilon$ this implies that $\int \cos(tx) d\mathbb{Q}(x) = 1$ and since $\cos(s) \leq 1$ this can happen only if

$$\mathbb{Q}\Big(\big\{x : xt = 0 \text{ mod } 2\pi\big\}\Big) = 1 \text{ for all } |t| \leq \varepsilon.$$

Take s, t such that $|s|, |t| \leq \varepsilon$ and s/t is irrational. For x to be in the support of $\mathbb{Q}$ we must have $xs = 2\pi k$ and $xt = 2\pi m$ for some integer k, m. This can happen only if $x = 0$.

□

Theorem 26 *(Levy's equivalence) If* (X_i) *is a sequence of independent r.v. then* $\sum_{i \geq 1} X_i$ *converges a.s. iff in probability iff in law.*

Proof. We already proved (a Kolmogorov's theorem) that convergence in probability implies a.s. convergence. It remains to prove only that convergence in law implies convergence in probability.

Suppose that $\mathcal{L}(S_n) \to \mathbb{P}$. Convergence in law implies that $\{\mathcal{L}(S_n)\}$ is uniformly tight which easily implies that $\{\mathcal{L}(S_n - S_k)\}_{n,k \leq 1}$ is uniformly tight. This will imply that for any $\varepsilon > 0$

$$\mathbb{P}(|S_n - S_k| > \varepsilon) < \varepsilon \tag{11.0.5}$$

for $n \geq k \geq N$ for large enough N. Suppose not. Then there exists $\varepsilon > 0$ and sequences $(n(l))$ and $(n'(l))$ such that $n(l) \leq n'(l)$ and

$$\mathbb{P}(|S_{n'(l)} - S_{n(l)}| > \varepsilon) \geq \varepsilon.$$

Let us denote $Y_l = S_{n'(l)} - S_{n(l)}$. Since $\{\mathcal{L}(Y_l)\}$ is uniformly tight, by selection theorem, there exists a subsequence $(l(r))$ such that $\mathcal{L}(Y_{l(r)}) \to \mathbb{Q}$. Since

$$S_{n'(l(r))} = S_{n(l(r))} + Y_{l(r)} \quad \text{and} \quad \mathcal{L}(S_{n'(l(r))}) = \mathcal{L}(S_{n(l(r))}) * \mathcal{L}(Y_{l(r)})$$

letting $r \to \infty$ we get that $\mathbb{P} = \mathbb{P} * \mathbb{Q}$. By the above Lemma, $\mathbb{Q}(\{0\}) = 1$, which implies that $\mathbb{P}(|Y_{l(r)}| > \varepsilon) \leq \varepsilon$ for large r - a contradiction. Once (11.0.5) is proved, by Borel-Cantelli lemma we can choose a.s. converging subsequence as in Kolmogorov's theorem, and then by (11.0.5) S_n converges in probability to the same limit.

□

Theorem 27 *(Three series theorem) Let* $(X_i)_{i \geq 1}$ *be a sequence of independent r.v. and let*

$$Z_i = X_i \mathrm{I}(|X_i| \leq 1).$$

Then $\sum_{i \geq 1} X_i$ *converges iff*

1. $\sum_{i \geq 1} \mathbb{P}(|X_i| > 1) < \infty$,
2. $\sum_{i \geq 1} \mathbb{E} Z_i$ *converges,*
3. $\sum_{i \geq 1} \mathrm{Var}(Z_i) < \infty$.

Proof. "$\Longleftarrow$". Suppose 1 - 3 hold. Since

$$\sum_{i \geq 1} \mathbb{P}(|X_i| > 1) = \sum_{i \geq 1} \mathbb{P}(X_i \neq Z_i) < \infty$$

by Borel-Cantelli lemma $\mathbb{P}(\{X_i \neq Z_i\} \text{ i.o.}) = 0$ which means that $\sum_{i\geq 1} X_i$ converges iff $\sum_{i\geq 1} Z_i$ converges. By 2, it is enough to show that $\sum_{i\geq 1}(Z_i - \mathbb{E}Z_i)$ converges, but this follows from Theorem 13 by 3.

"$\Longrightarrow$". If $\sum_{i\geq 1} X_i$ converges a.s.,

$$\mathbb{P}(\{|X_i| > 1\} \text{ i.o}) = 0$$

and since (X_i) are independent, by Borel-Cantelli,

$$\sum_{i\geq 1} \mathbb{P}(|X_i| > 1) < \infty.$$

If $\sum_{i\geq 1} X_i$ converges then, obviously, $\sum_{i\geq 1} Z_i$ converges. Let

$$S_{mn} = \sum_{m\leq k\leq n} Z_k.$$

Since $S_{mn} \to 0$ as $m, n \to \infty$, $\mathbb{P}(|S_{mn}| > \delta) \leq \delta$ for any $\delta > 0$ for m, n large enough. Suppose that $\sum_{i\geq 1} \mathrm{Var}(Z_i) = \infty$. Then

$$\sigma^2_{mn} = \mathrm{Var}(S_{mn}) = \sum_{m\leq k\leq n} \mathrm{Var}(Z_k) \to \infty$$

as $n \to \infty$ for any fixed m. Intuitively, this should not happen: $S_{mn} \to 0$ in probability but their variance goes to infinity. In principle, one can construct such sequence of random variables but in our case it will be ruled out by Lindeberg's CLT. Because $\sigma_{mn} \to \infty$, Lindeberg's theorem will imply that,

$$T_{mn} = \frac{S_{mn} - \mathbb{E}S_{mn}}{\sigma_{mn}} = \sum_{m\leq k\leq n} \frac{Z_k - \mathbb{E}Z_k}{\sigma_{mn}} \to \mathcal{N}(0,1),$$

if $m, n \to \infty$ and $\sigma^2_{mn} \to \infty$. We only need to check that

$$\sum_{m\leq k\leq n} \mathbb{E}\Big(\frac{Z_k - \mathbb{E}Z_k}{\sigma_{mn}}\Big)^2 \mathrm{I}\Big(\ \Big|\frac{Z_k - \mathbb{E}Z_k}{\sigma_{mn}}\Big| > \varepsilon\Big) \to 0$$

as $m, n, n-m \to \infty$. Since $|Z_k - \mathbb{E}Z_k| < 2$ and $\sigma_{mn} \to \infty$, the event in the indicator does not occur for large m, n and, therefore, the sum is equal to 0. Next, since $\mathbb{P}(|S_{mn}| \leq \delta) \geq 1 - \delta$ we get

$$P\Big(\Big|T_{mn} + \frac{\mathbb{E}S_{mn}}{\sigma_{mn}}\Big| \leq \frac{\delta}{\sigma_{mn}}\Big) \geq 1 - \delta.$$

But this is impossible, since T_{mn} is approximately standard normal and standard normal distribution does not concentrate near any constant. We proved that $\sum_{i\geq 1} \mathrm{Var}(Z_i) < \infty$. By Kolmogorov's SLLN this implies that $\sum_{i\geq 1}(Z_i - \mathbb{E}Z_i)$ converges and, therefore, $\sum_{i\geq 1} \mathbb{E}Z_i$ converges.

□